On the Density of Sets Avoiding Parallelohedron Distance 1

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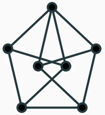
Computational Challenges in the Theory of Lattices - 26.04.2018

Introduction

Warm up: reminder about graphs

- Let G be a graph, that is a set of vertices V and a set of edges $E \subset V^2$.



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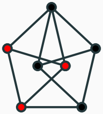

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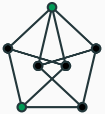

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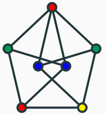

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- The chromatic number $\chi(G)$ of G is the least number of colors required to color V in such a way that two neighbors do not receive the same color.

The Unit Distance Graph

- The unit distance graph $G(\mathbb{R}^n, \|\cdot\|)$:
  - Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.
  - The vertices are the points of $\mathbb{R}^n$,
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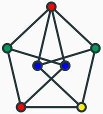
- The Hadwiger-Nelson problem (1950): What is the chromatic number $\chi(\mathbb{R}^2)$ of the unit distance graph for the Euclidean norm?
- We also define the measurable chromatic number $\chi_m(\mathbb{R}^n)$, when we assume that the color classes are measurable.

Known results about $\chi(\mathbb{R}^2)$ and $\chi(\mathbb{R}^n)$

There is a coloring of the Euclidean plane with seven colors:

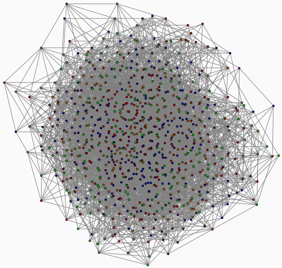
The Moser graph has chromatic number 4:



So $4 \leq \chi(\mathbb{R}^2) \leq 7$.

Known results about $\chi(\mathbb{R}^2)$ and $\chi(\mathbb{R}^n)$

April 2018, the 5 shades of de Grey:



So $5 \leq \chi(\mathbb{R}^2) \leq 7$.
Already known $5 \leq \chi_m(\mathbb{R}^2) \leq 7$ (Falconer 1981).

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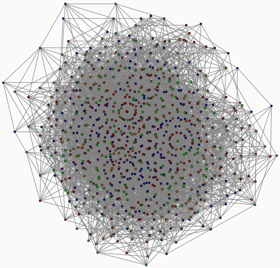

So $5 \leq \chi(\mathbb{R}^2) \leq 7$.
Asymptotically:

$$(1.2)^n \lessapprox \chi(\mathbb{R}^n) \lessapprox 3^n$$

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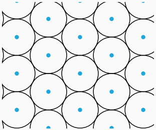
- We have the relation

$$\chi_m(\mathbb{R}^n, \|\cdot\|) \geq \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}.$$

Constructions using packings

- Let Λ be a packing of balls of radius 1 in $\mathbb{R}^n$:



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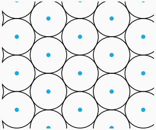
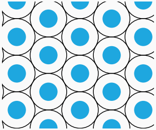


- Then the set $A = \bigcup_{\lambda \in \Lambda} B(\lambda, 1/2)$ avoids 1:



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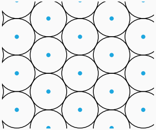

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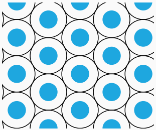

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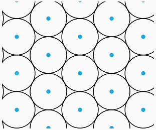

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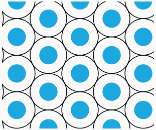

- The density of A is then $\delta(A) = \frac{\Delta(\Lambda)}{2^n}$, where $\Delta(\Lambda)$ is the density of the packing.

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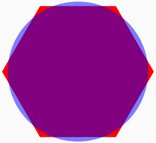

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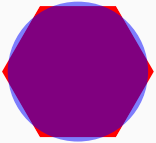

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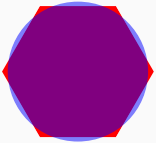

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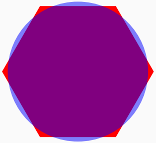

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- Conjecture (Moser, Larman, Rogers): $m_1(\mathbb{R}^n, \| \cdot \|_2) < \frac{1}{2^n}$.

Our problem

Let $\|\cdot\|_{\mathcal{P}}$ be a norm on $\mathbb{R}^n$ such that the unit ball is a polytope $\mathcal{P}$ that tiles $\mathbb{R}^n$ by translation, then

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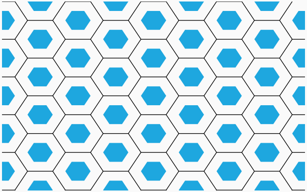

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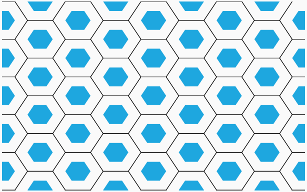

Conjecture (Bachoc, Robins)

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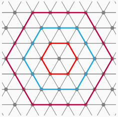

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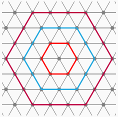

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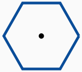
- Example: 

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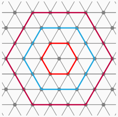

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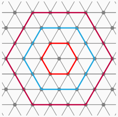

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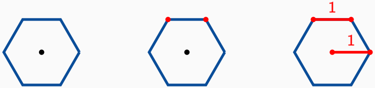

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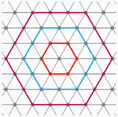

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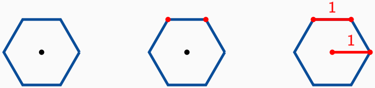
- Example: 

- The norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are polytope norms.

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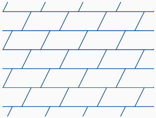

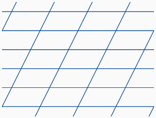

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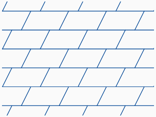
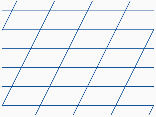
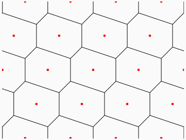




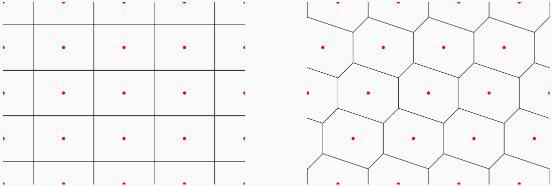
- The convex polytopes that tile $\mathbb{R}^n$ by translation are the parallelohedra (Venkov, 1954).
- Voronoi conjecture, 1908: every parallelohedron is affinely equivalent to the Voronoi region of a lattice. True for dimensions $n \leq 4$ (Delone, 1929).



First Approach

Dimension 2

There are two kinds of Voronoi regions in $\mathbb{R}^2$:



Theorem (Bachoc, Bellitto, M., Pêcher)

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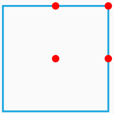

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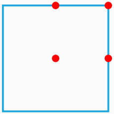

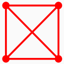

- This subgraph is a complete graph with 4 vertices.

Strategy

- Let $G = (V, E)$ a finite induced subgraph of the unit distance graph. Let $\alpha(G)$ be its independence number.
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$$m_1(\mathbb{R}^n, \|\cdot\|) \leq \frac{\alpha(G)}{|V|}$$

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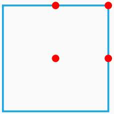

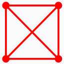

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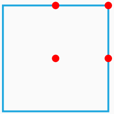

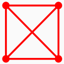

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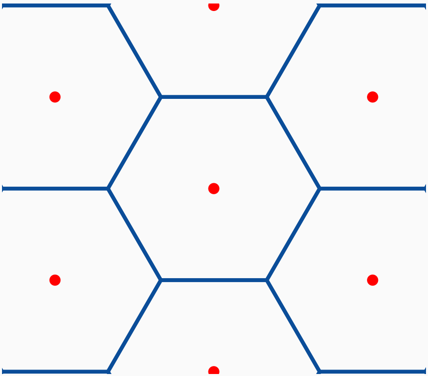
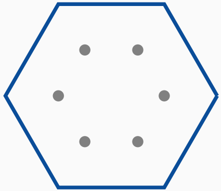
- This inequality can be extended to discrete subgraphs.

The regular hexagon (Dmitry Shiryaev)

The regular hexagon $\mathcal{H}_0$ is the Voronoi region of the hexagonal lattice A_2. .

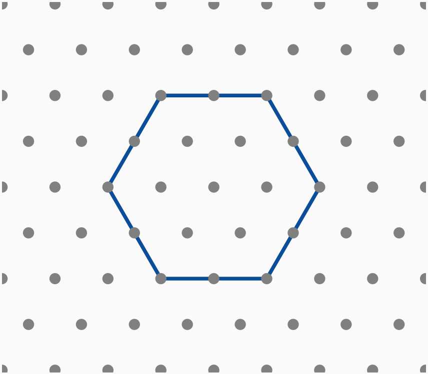


Let S be the set of vertices of $\mathcal{H}_0$. Consider the set $\frac{1}{2}S$.



The regular hexagon (Dmitry Shiryaev)

. The set $\frac{1}{2}S$ spans the lattice $V = \frac{1}{2}A_2^{\#}$. We consider the subgraph G of $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}_0})$ induced by V.



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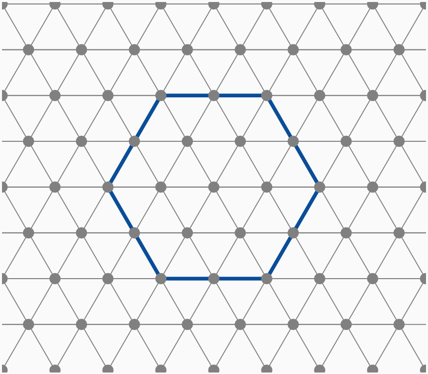
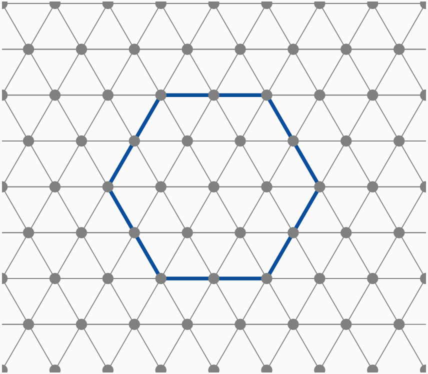
. We consider the auxiliary graph $\tilde{G}$ whose set of vertices is V and whose edges are the pairs $\{x, y\}$ such that $x - y \in \frac{1}{2}S$.



The regular hexagon (Dmitry Shiryaev)

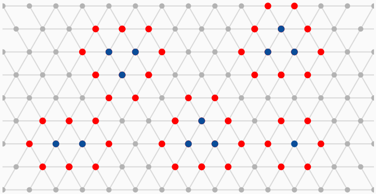
- We have:

$$d_{\tilde{G}}(x, y) = 2 \Leftrightarrow ||x - y||_{\mathcal{P}} = 1.$$



The regular hexagon

- So a set avoiding hexagon distance 1 in V must be the union of cliques in $\tilde{G}$ whose closed neighborhoods must be disjoint.



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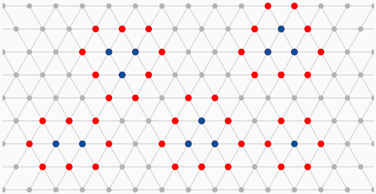
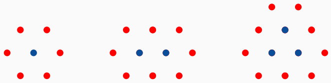


- So the density of a set avoiding hexagon distance 1 cannot be better than:



$\frac{1}{7}$ $\qquad$ $\frac{2}{10} = \frac{1}{5}$ $\qquad$ $\frac{3}{12} = \frac{1}{4}$

The general case

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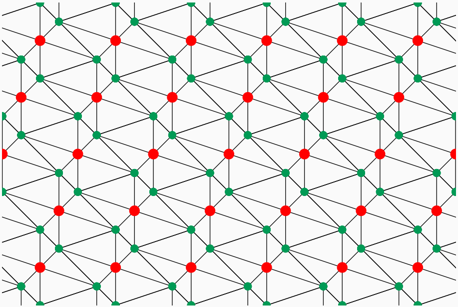
- The previous construction using $\frac{1}{2}$-vertices does not work anymore.
- By considering another graph, we can prove that $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}}) = \frac{1}{4}$.



Infinite families of lattices

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- The lattice $A_n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_1^{n+1} x_i = 0\}$:

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- The lattice $D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum_1^n x_i = 0 \mod 2\}$:

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- For Λ in those two families, the vertices of the Voronoi region of Λ span $\Lambda^\#$.
- In both cases, we have an auxiliary graph $\tilde{G}$ such that

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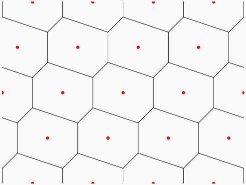
How to generalize this method?

Discrete Distribution Functions

Another graph in dimension 2

Let L be a lattice in $\mathbb{R}^2$.



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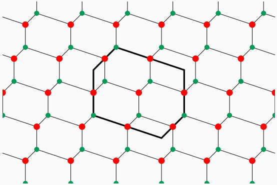
Consider the following graph. The set of vertices is $\frac{1}{2}L \cup \left(v_1 + \frac{1}{2}L\right)$



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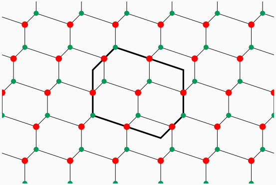

The auxiliary graph satisfies $d_{\tilde{G}}(x, y) = 2 \Rightarrow \|x - y\|_{\mathcal{P}} = 1$.

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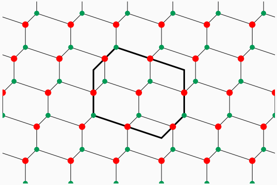
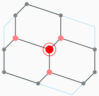
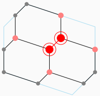


Thus we can apply the previous method. There are two kinds of cliques:





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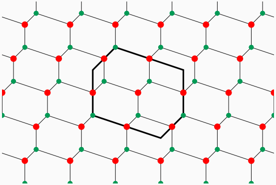

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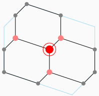

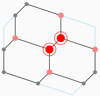

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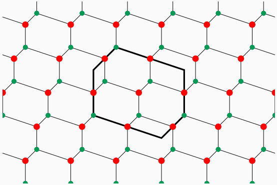

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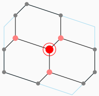
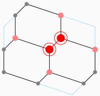
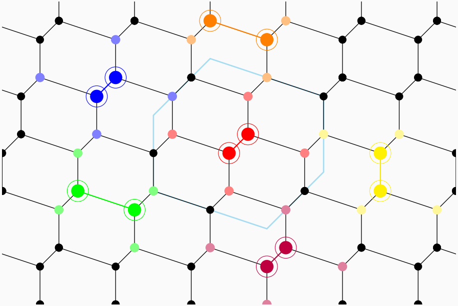




The upper bound obtained is $1/3$. Pretty bad...

Another graph in dimension 2

The closed neighborhoods of the cliques cannot fill the whole set of vertices.



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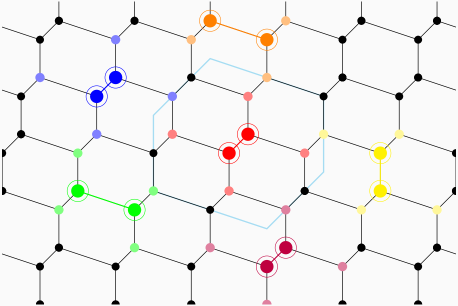

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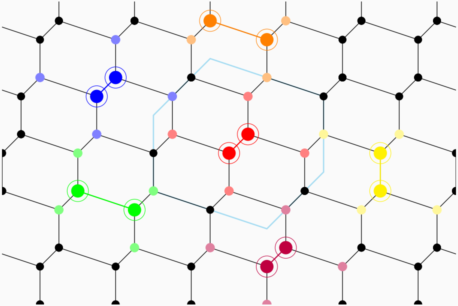

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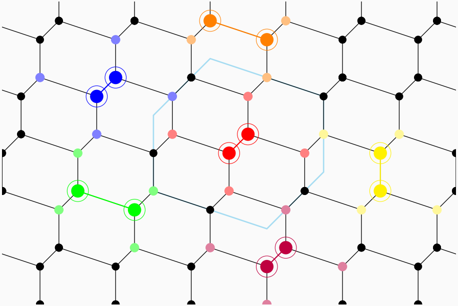

What to do to with the free points? We have to chose how to distribute the vertices of V among the cliques. Every clique will be given a new neighborhood.

Another graph in dimension 2

For every clique C, every vertex $x \in V$ such that $d_{\mathcal{P}}(x, C) \leq 1$ will contribute to the neighborhood of C:

red points: 1 orange points: 2/3 yellow points: 1/3

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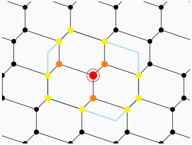

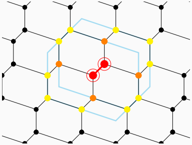

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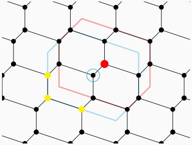
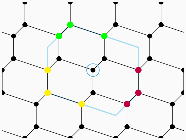
We check that the total contribution of a vertex x is at most 1.





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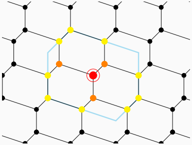

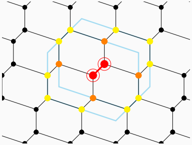

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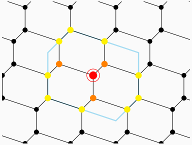

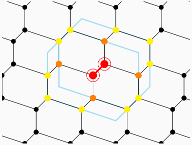

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We used a discrete distribution function.

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- We have a discrete distribution function $f : (x, C) \to f(x, C) \in [0, 1]$ such that $\forall x \in V,\ \sum_{\mathcal{C}} f(x, C) \leq 1$.

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- We have a discrete distribution function $f : (x, C) \to f(x, C) \in [0, 1]$ such that $\forall x \in V,\ \sum_{\mathcal{C}} f(x, C) \leq 1$.

Then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leq \sup_C \frac{|C|}{|Ne(C)|}$.

Constraints

If we want to apply this method to an induced subgraph $G = (V, E)$ of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$, we need an auxiliary graph $\tilde{G}$ such that:

- The set of vertices of $\tilde{G}$ is also V.
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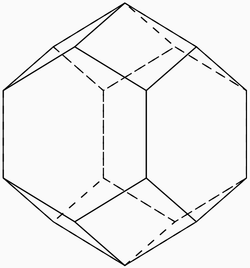
Then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leq \sup_C \frac{|C|}{\sum_{x \in V} f(x,C)}$.

Results

With this method, we show:

Theorem (M.)

If $\mathcal{P}$ is the Voronoi region of the lattice L spanned by $\mathcal{B} = \{(2,0,0),(0,2,0),(-1,-1,2)\}$, then $m_1(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}}) = \frac{1}{8}$.



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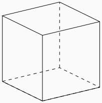

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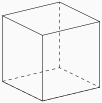

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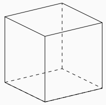

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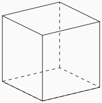

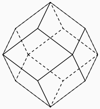

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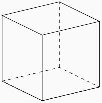

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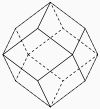

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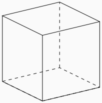

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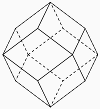

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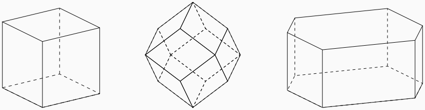

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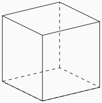
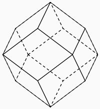
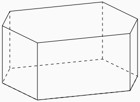

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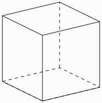

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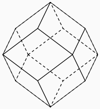

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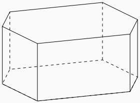

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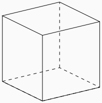 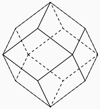 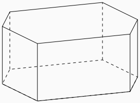 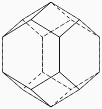

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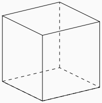

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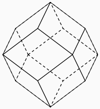

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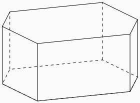

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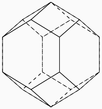

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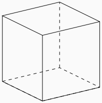

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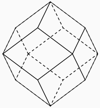

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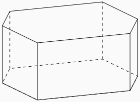

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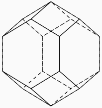

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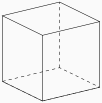

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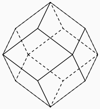

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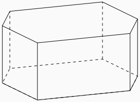

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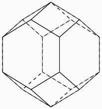

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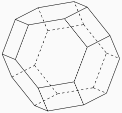

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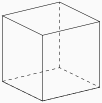

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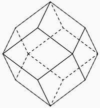

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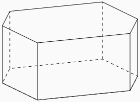

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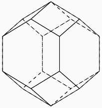

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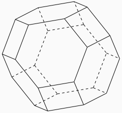

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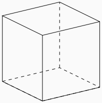

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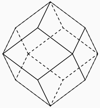

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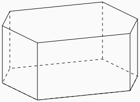

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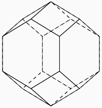

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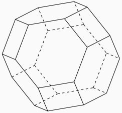

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???

Ongoing work and perspectives

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If $\mathcal{P}$ is the Voronoi cell of the lattice $\pi^{-1}(C_n)$ where $C_n \subset \mathbb{F}_2^n$ has minimal distance at least αn, then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leq 2^{-\lfloor \alpha n/2 \rfloor}$.

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If $\mathcal{P}$ is the Voronoi cell of $D_n^{\#}$ then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leq 2^{-\lfloor n/2 \rfloor}$.

If $\mathcal{P}$ is the Voronoi cell of D_{2k}^{+} then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leq 2^{-k}(4/3 + o(1))$.

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Thank you for your attention!